CSE 250
Data Structures

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Day 23
Priority Queues, Heaps
Textbook Ch. 18
Announcements

- WA2 released, due 10/31
- PA3 has not been released yet
- Midterm exam grades have been posted
PriorityQueue ADT

PriorityQueue[A <: Ordering]

enqueue(v: A): Unit
   Insert value v into the priority queue

dequeue: A
   Remove the greatest element in the priority queue

head: A
   Peek at the greatest element in the priority queue
Priority Queues

Two mentalities...

Lazy: Keep everything a mess ("Selection Sort")

Proactive: Keep everything organized ("Insertion Sort")
## Priority Queues

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## Priority Queues

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Can we do better?
Priority Queues

Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue
Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue

??? - Fast(-ish) Enqueue, Fast(-ish) Dequeue
**Priority Queues**

*Idea:* Keep the priority queue "kinda" sorted. Hopefully "kinda" sorted is cheaper to maintain than a full sort, but still gives us some of the benefits.
Idea: Keep the priority queue "kinda" sorted.

Keep larger items towards the front of the list, and keep the front of the list more sorted than the back...
Binary Heaps

**Challenge:** If we are only "kinda" sorting, how do we know which elements are actually sorted?
**Binary Heaps**

**Idea:** Organize the priority queue as a *directed* tree!

A directed edge from $a$ to $b$ means that $a \geq b$.
More Tree Terminology

**Child** - An adjacent node connected by an out-edge
More Tree Terminology

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**Leaf** - A node with no children
More Tree Terminology

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**Depth (of a node)** - The number of edges from the root to the node
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**Depth (of a tree)** - The maximum depth of any node in the tree
More Tree Terminology

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**Leaf** - A node with no children

**Depth (of a node)** - The number of edges from the root to the node

**Depth (of a tree)** - The maximum depth of any node in the tree

**Level (of a node)** - depth + 1
More Tree Terminology

A is the root

B and C are children of A
D is a child of C
E and F are children of D

B, E and F are leaves

The depth of A is 0, B and C: 1, D: 2, E and F: 3

The depth of the tree is 3
Organize our priority queue as a directed tree

**Directed:** A directed edge from $a$ to $b$ means that $a \geq b$
Binary Heaps

Organize our priority queue as a directed tree

**Directed:** A directed edge from $a$ to $b$ means that $a \geq b$

**Binary:** Max out-degree of 2 (easy to reason about)
Binary Heaps

Organize our priority queue as a directed tree

**Directed:** A directed edge from \( a \) to \( b \) means that \( a \geq b \)

**Binary:** Max out-degree of 2 (easy to reason about)

**Complete:** Every "level" except the last is full (from left to right)
Binary Heaps

Organize our priority queue as a directed tree

Directed: A directed edge from $a$ to $b$ means that $a \geq b$

Binary: Max out-degree of 2 (easy to reason about)

Complete: Every "level" except the last is full (from left to right)

Balanced: TBD (basically, all leaves are roughly at the same level)
Binary Heaps

Organize our priority queue as a directed tree

**Directed:** A directed edge from $a$ to $b$ means that $a \geq b$

**Binary:** Max out-degree of 2 (easy to reason about)

**Complete:** Every "level" except the last is full (from left to right)

**Balanced:** TBD (basically, all leaves are roughly at the same level)

*This makes it easy to encode into an array (later today)*
Max Heaps

If we use $\geq$ as our ordering operation, we have a Max Heap
(as compared to a Min Heap)
Valid Max Heaps
Invalid Max Heaps
Invalid Max Heaps

Need to fill from left to right
Invalid Max Heaps

Need to fill from left to right

Need complete levels
Invalid Max Heaps

Need to fill from left to right

Need complete levels

Children must be less than or equal to parents
What is the depth of a binary heap containing $n$ items?

Level 1: holds up to 1 item
What is the depth of a binary heap containing $n$ items?

**Level 1:** holds up to 1 item

**Level 2:** holds up to 2 items
What is the depth of a binary heap containing \( n \) items?

- **Level 1**: holds up to 1 item
- **Level 2**: holds up to 2 items
- **Level 3**: holds up to 4 items
What is the depth of a binary heap containing \( n \) items?

- **Level 1**: holds up to 1 item
- **Level 2**: holds up to 2 items
- **Level 3**: holds up to 4 items
- **Level 4**: holds up to 8 items
What is the depth of a binary heap containing \( n \) items?

**Level 1:** holds up to 1 item

**Level 2:** holds up to 2 items

**Level 3:** holds up to 4 items

**Level 4:** holds up to 8 items

... 

**Level \( i \):** holds up to \( 2^{i-1} \) items
What is the depth of a binary heap containing \( n \) items?

\[
n = O \left( \sum_{i=1}^{l_{\text{max}}} 2^i \right) = O \left( 2^{l_{\text{max}}} \right)
\]
What is the depth of a binary heap containing $n$ items?

$$n = O \left( \sum_{i=1}^{\ell_{\text{max}}} 2^i \right) = O \left( 2^{\ell_{\text{max}}} \right)$$

$$\ell_{\text{max}} = O \left( \log(n) \right)$$
The Heap ADT

enqueue(elem: A): Unit  \[\text{AKA pushHeap}\]
Place an item into the heap

dequeue: A  \[\text{AKA popHeap}\]
Remove and return the maximal element from the heap

head: A
Peek at the maximal element in the heap

length: Int
The number of elements in the heap
Idea: Insert the element at the next available spot, then fix the heap.
**Heap.enqueue**

**Idea:** Insert the element at the next available spot, then fix the heap.

1. Call the insertion point `current`
2. While `current != root` and `current > parent`
   a. Swap `current` with `parent`
   b. Repeat with `current ← parent`
Heap.enqueue

What if we enqueue 6?
Heap.enqueue

What if we enqueue 6?
Place in the next available spot
What if we enqueue 6?
Swap with parent if it is bigger than the parent
Heap.enqueue

What if we enqueue 6?
Continue swapping upwards...
Heap.enqueue

What if we enqueue 6?
Stop swapping when we are no longer bigger than our parent.
def fixUp[A: Ordering](current: Vertex[A]): Unit = {
  if(current.parent.isDefined) {
    val parent = current.parent.get
    if (Ordering[A].lt(parent.value, current.value)) {
      swap(current.value, parent.value)
      fixUp(parent)
    }
  }
}
def fixUp[A: Ordering](current: Vertex[A]): Unit = {
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    if (Ordering[A].lt( parent.value, current.value ) ) {
      swap(current.value, parent.value)
      fixUp(parent)
    }
  }
}
What is the complexity (or how many swaps occur)? $O(\log(n))$
Heap.dequeue
Heap.dequeue

Idea: Replace root with the last element then fix the heap
**Algorithm:** Heap.dequeue

**Idea:** Replace root with the last element then fix the heap

1. Start with \( \text{current} \leftarrow \text{root} \)
2. While \( \text{current} \) has a child > \( \text{current} \)
   a. Swap \( \text{current} \) with its largest child
   b. Repeat with \( \text{current} \leftarrow \text{child} \)
Heap. dequeue

What if we call dequeue?
What if we call dequeue?
Remove and return the root
What if we call dequeue?
Make the last item the new root
What if we call dequeue?
Check for our largest child
Heap.dequeue

What if we call dequeue?
If the largest child is bigger than us, swap
Heap::dequeue

What if we call dequeue?
Continue swapping down the tree as necessary...
What if we call dequeue?
Continue swapping down the tree as necessary...
What if we call dequeue?

Stop swapping when our children are no longer bigger
Heap.dequeue - fixDown

```scala
def fixDown[A: Ordering](current: Vertex[A]): Unit = {
  val maxChild = getMaxChildOf(current)
  if (maxChild.isDefined) {
    val max = maxChild.get
    if (Ordering[A].lt(current.value, max.value)) {
      swap(current.value, max.value);
      fixDown(max)
    }
  }
}
```
Heap.dequeue - fixDown

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def fixDown[A: Ordering](current: Vertex[A]): Unit = {
val maxChild = getMaxChildOf(current)
if( maxChild.isDefined ) {
  val max = maxChild.get
  if( Ordering[A].lt( current.value, max.value ) ){
    swap(current.value, max.value);
    fixDown(max)
  }
}
}
```

What is the complexity (or how many swaps occur)?
Heap.dequeue - fixDown

What is the complexity (or how many swaps occur)? $O(\log(n))$
# Priority Queues

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Notice that:

1. Each level has a maximum size
2. Each level grows left-to-right
3. Only the last layer grows

How can we compactly store a heap?
Storing heaps

Notice that:
1. Each level has a maximum size
2. Each level grows left-to-right
3. Only the last layer grows

How can we compactly store a heap?

Idea: Use an ArrayBuffer
Storing Heaps

How can we store this heap in an array buffer?
Storing Heaps

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How can we store this heap in an array buffer?

Enqueue always inserts at the arrays end (then we fix up)
Runtime Analysis

enqueue

- Append to ArrayBuffer:
  - Amortized $O(1)$ (worst-case $O(n)$)
- fixUp:
  - $O(\log(n))$ fixes, each one costs $O(1) = O(n \log(n))$
- Total: Amortized $O(n \log(n))$ (worst-case $O(n)$)

dequeue

- Remove end of ArrayBuffer:
  - $O(1)$
- fixDown:
  - $O(\log(n))$ fixes, each one costs $O(1) = O(n \log(n))$
- Total: Worst-case $O(n \log(n))$
Runtime Analysis

enqueue

- **Append to ArrayBuffer**: amortized $O(1)$ (worst-case $O(n)$)
Runtime Analysis

**enqueue**
- **Append to ArrayBuffer**: amortized $O(1)$ (worst-case $O(n)$)
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dequeue

- **Remove end of ArrayBuffer**: $O(1)$
Runtime Analysis

**enqueue**
- **Append to ArrayBuffer**: amortized $O(1)$ *(worst-case $O(n)$)*
- **fixUp**: $O(\log(n))$ fixes, each one costs $O(1) = O(\log(n))$
- **Total**: amortized $O(\log(n))$ *(worst-case $O(n)$)*

**dequeue**
- **Remove end of ArrayBuffer**: $O(1)$
- **fixDown**: $O(\log(n))$ fixes, each one costs $O(1) = O(\log(n))$
Runtime Analysis

enqueue
- **Append to ArrayBuffer**: amortized $O(1)$ (worst-case $O(n)$)
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dequeue
- **Remove end of ArrayBuffer**: $O(1)$
- **fixDown**: $O(\log(n))$ fixes, each one costs $O(1) = O(\log(n))$
- **Total**: worst-case $O(\log(n))$
Heap Sort

1. Insert items into heap
2. Reconstruct sequence (in reverse order) with dequeue

7, 4, 8, 2, 5, 3, 9
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Heap Sort
Heap Sort

Enqueue element $i$: $O(\log(i))$
Heap Sort

Enqueue element $i$: $O(\log(i))$

Dequeue element $i$: $O(\log(n - i))$
Heap Sort

**Enqueue element** $i$: $O(\log(i))$

**Dequeue element** $i$: $O(\log(n - i))$

$$
\left( \sum_{i=1}^{n} O(\log(i)) \right) + \left( \sum_{i=1}^{n} O(\log(n - i)) \right)
$$
Heap Sort

Enqueue element \(i\): \(O(\log(i))\)

Dequeue element \(i\): \(O(\log(n - i))\)

\[
\left( \sum_{i=1}^{n} O(\log(i)) \right) + \left( \sum_{i=1}^{n} O(\log(n - i)) \right) < O(n \log(n))
\]
What if we want to update a value in our Heap?
What if we want to update a value in our Heap?

After update we can just call `fixUp` or `fixDown` based on the new value.
What if we want to update a value in our Heap?

After update we can just call `fixUp` or `fixDown` based on the new value

*Can we apply this idea to an entire array?*
Heapify

**Input:** Array

**Output:** Array re-ordered to be a heap
Heapify
Heapify

At level $\log(n)$: Call $\text{fixDown}$ on all $n/2$ nodes at this level (max 0 swaps each)
Heapify

At level $\log(n)$: Call $\text{fixDown}$ on all $n/2$ nodes at this level (max 0 swaps each)

At level $\log(n)-1$: Call $\text{fixDown}$ on all $n/4$ nodes at this level (max 1 swaps each)
Heapify

At level $\log(n)$: Call fixDown on all $n/2$ nodes at this level (max 0 swaps each)

At level $\log(n)-1$: Call fixDown on all $n/4$ nodes at this level (max 1 swaps each)

At level $\log(n)-2$: Call fixDown on all $n/8$ nodes at this level (max 2 swaps each)
Heapify

At level $\log(n)$: Call $\text{fixDown}$ on all $n/2$ nodes at this level (max 0 swaps each)

At level $\log(n)-1$: Call $\text{fixDown}$ on all $n/4$ nodes at this level (max 1 swaps each)

At level $\log(n)-2$: Call $\text{fixDown}$ on all $n/8$ nodes at this level (max 2 swaps each)

... 

At level 1: Call $\text{fixDown}$ on all 1 nodes at this level (max $\log(n)$ swaps each)
Heapify

Sum the number of swaps required by each level

\[
O \left( \sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)
\]
Heapify

Pull out the $n$ as a constant and distribute multiplication
Heapify

Focus on the dominant term only
Heapify

Change $\log(n)$ to infinity (can only increase complexity class if anything)
Heapify

We can now treat the sum as a constant.

This is known to converge to a constant.
Heapify

Therefore we can heapify an array of size $n$ in $O(n)$.
Heapify

Therefore we can heapify an array of size $n$ in $O(n)$ (but heap sort still requires $n \log(n)$ due to dequeue costs)
Next time...

Sets, Bags, and Binary Search Trees...