Announcements

- WA1 due tonight at 11:59PM
  - Late submissions (up to tomorrow at 11:59PM) receive 50% penalty
- PA2 is released
  - Start early......please :)


Recap - Merge Sort

**Divide:** Split the sequence in half
\[ D(n) = \Theta(n) \text{ (can do in } \Theta(1)) \]

**Conquer:** Sort the left and right halves
\[ a = 2, \ b = 2, \ c = 1 \]

**Combine:** Merge halves together
\[ C(n) = \Theta(n) \]
Merge Sort: Intuition
Merge Sort: Intuition

Each time we move down a level, we split the sequence in half.

\[ \Theta(n) \to \Theta(n/2) \to \Theta(n/4) \to \Theta(n/4) \to \Theta(n/4) \to \Theta(n/4) \]
Merge Sort: Intuition

Each time we move down a level, we split the sequence in half.

Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels.
Merge Sort: Intuition

- Each time we move down a level, we split the sequence in half.

- Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels.

- Notice the total cost of each level is always $\Theta(n)$. 
Merge Sort: Intuition

Because we divide in half at each level, we have \( \log(n) \) levels.

Each time we move down a level, we split the sequence in half.

Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels.

Notice the total cost of each level is always \( \Theta(n) \).
Merge Sort: Intuition

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Each time we move down a level, we split the sequence in half.

Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels.

Notice the total cost of each level is always \( \Theta(n) \).

Hypothesis: The cost of merge sort is \( n \log(n) \).
Base Case: $T(1) \leq c$

$c_0 \leq c$

True for any $c > c_0$
Merge Sort: Proof by Induction

Assume: $T(n/2) \leq c \frac{n}{2} \log(n/2)$

Show: $T(n) \leq cn \log(n)$
Merge Sort: Proof by Induction

Assume: \( T(n/2) \leq c \frac{n}{2} \log(n/2) \)

Show: \( T(n) \leq cn \log(n) \)

\[
2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)
\]
Merge Sort: Proof by Induction

**Assume:** \( T(n/2) \leq c \frac{n}{2} \log(n/2) \)

**Show:** \( T(n) \leq cn \log(n) \)

\[
2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)
\]

By the assumption, and transitivity, we just need to show:

\[
2c \frac{n}{2} \log \left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)
\]
Merge Sort: Proof by Induction

Assume: \( T(n/2) \leq c (n/2) \log(n/2) \)

Show: \( T(n) \leq cn \log(n) \)

\[
2 \cdot T\left( \frac{n}{2} \right) + c_1 + c_2 n \leq cn \log(n)
\]

By the assumption, and transitivity, we just need to show:

\[
2c \frac{n}{2} \log \left( \frac{n}{2} \right) + c_1 + c_2 n \leq cn \log(n)
\]

\[
 cn \log(n) - cn \log(2) + c_1 + c_2 n \leq cn \log(n)
\]
Merge Sort: Proof by Induction

**Assume:** $T(n/2) \leq c \ (n/2) \log(n/2)$

**Show:** $T(n) \leq cn \log(n)$

$$2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2 n \leq cn \log(n)$$

By the assumption, and transitivity, we just need to show:

$$2c \cdot \frac{n}{2} \log \left(\frac{n}{2}\right) + c_1 + c_2 n \leq cn \log(n)$$

$$cn \log(n) - cn \log(2) + c_1 + c_2 n \leq cn \log(n)$$

$$c_1 + c_2 n \leq cn \log(2)$$
Merge Sort: Proof by Induction

\[ c_1 + c_2 n \leq cn \log(2) \]
Merge Sort: Proof by Induction

\[ c_1 + c_2 n \leq cn \log(2) \]

\[ \frac{c_1}{n \log(2)} + \frac{c_2}{\log(2)} \leq c \]
Merge Sort: Proof by Induction

Which is true for any and
Where is all of the "work" being done?
Where is all of the "work" being done?

The combine step
Where is all of the "work" being done?

**The combine step**

Can we put the work in the divide step instead?
QuickSort

**Idea**: What if we divide our sequence around a particular value?

What value would we like to choose?
Idea: What if we divide our sequence around a particular value?

What value would we like to choose? Median
QuickSort: Idealized Version

7  1  4  3  5  2  6  8
QuickSort: Idealized Version

7  1  4  3  |  5  2  6  8
QuickSort: Idealized Version

7  1  4  3  5  2  6  8

2  1  4  3  5  7  6  8
QuickSort: Idealized Version

Here is a diagram of the QuickSort algorithm in its idealized version.
QuickSort: Idealized Version

7  1  4  3  5  2  6  8
2  1  4  3  5  7  6  8
QuickSort: Idealized Version
QuickSort: Idealized Version

```
7 1 4 3 | 5 2 6 8
2 1 4 3 | 5 7 6 8
1 2 4 3 | 5 7 6 8
1 2 3 4 | 5 7 6 8
```
QuickSort: Idealized Version

1 2 3 4 5 7 6 8
QuickSort: Idealized Version

[Diagram showing the sorting process]

7  1  4  3  5  2  6  8
2  1  4  3  5  7  6  8
1  2  4  3  5  7  6  8
1  2  3  4  5  7  6  8
1  2  3  4  5  6  7  8
QuickSort: Idealized Version

7  1  4  3  5  2  6  8

2  1  4  3  5  7  6  8

1  2  4  3  5  7  6  8

1  2  3  4  5  7  6  8

1  2  3  4  5  6  7  8
QuickSort: Idealized Version
QuickSort: Idealized Version
QuickSort: Idealized Algorithm

To sort an array of size $n$:

1. Pick a pivot value (median?)
2. Swap values until:
   a. elements at $[1, n/2)$ are $\leq$ pivot
   b. elements at $[n/2, n)$ are $>$ pivot
3. Recursively sort the lower half
4. Recursively sort the upper half
def idealizedQuickSort(arr: Array[Int], from: Int, until: Int): Unit = {
  if(until - from < 1) { return }
  val pivot = ???
  var low = from, high = until -1

  while(low < high) {
    while(arr(low) <= pivot && low < high){ low ++ }
    if(low < high) {
      while(arr(high) > pivot && low < high){ high ++ }
      swap(arr, low, high)
    }
  }

  idealizedQuickSort(arr, from = 0, until = low)
  idealizedQuickSort(arr, from = low, until = until)
}
Great! So...how do we find the median...?
Great! So...how do we find the median...?

Finding the median takes $O(n \log(n))$ for an unsorted array :(
Imagine a world where we can obtain a pivot in $O(1)$. Now what is our complexity?
QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in $O(1)$. Now what is our complexity?

$$T_{\text{quick sort}}(n) = \begin{cases} 
\Theta(1) & \text{if } n = 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(n) + 0 & \text{otherwise}
\end{cases}$$
QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in $O(1)$. Now what is our complexity?

$$T_{\text{quicksort}}(n) = \begin{cases} 
\Theta(1) & \text{if } n = 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(n) + 0 & \text{otherwise}
\end{cases}$$

Compare to Merge Sort:

$$T_{\text{mergesort}}(n) = \begin{cases} 
\Theta(1) & \text{if } n = 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases}$$
QuickSort: Attempt #2

So how can we pick a pivot value (in O(1) time)?
QuickSort: Attempt #2

So how can we pick a pivot value (in O(1) time)?

**Idea:** Pick it randomly! On average, half the values will be lower.
To sort an array of size $n$:

1. Pick a value at random as the *pivot*
2. Swap values until the array is subdivided into:
   a. *low*: array elements $< \text{pivot}$
   b. *pivot*
   c. *high*: array elements $> \text{pivot}$
3. Recursively sort *low*
4. Recursively sort *high*
QuickSort: Runtime

What is the worst-case runtime?
QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

[8, 7, 6, 5, 4, 3, 2, 1]
QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

\[ [8,7,6,5,4,3,2,1] \]

\[ [7,6,5,4,3,2,1],8,[ ] \]
QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

[8, 7, 6, 5, 4, 3, 2, 1]
[7, 6, 5, 4, 3, 2, 1], 8, []
[6, 5, 4, 3, 2, 1], 7, [], 8
QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

\[ [8, 7, 6, 5, 4, 3, 2, 1] \]
\[ [7, 6, 5, 4, 3, 2, 1], 8, [ ] \]
\[ [6, 5, 4, 3, 2, 1], 7, [ ], 8 \]
\[ [5, 4, 3, 2, 1], 6, [ ], 7, 8 \]
QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

[8, 7, 6, 5, 4, 3, 2, 1]
[7, 6, 5, 4, 3, 2, 1], 8, []
[6, 5, 4, 3, 2, 1], 7, [], 8
[5, 4, 3, 2, 1], 6, [], 7, 8

...
QuickSort: Worst-Case Runtime

What is the worst-case runtime?
QuickSort: Worst-Case Runtime

What is the worst-case runtime?

\[ T_{\text{quicksort}}(n) \in O(n^2) \]
QuickSort: Worst-Case Runtime

Is the worst case runtime representative?
QuickSort: Worst-Case Runtime

Is the worst case runtime representative?

No! (the actual runtime will almost always be faster)
QuickSort: Worst-Case Runtime

Is the worst case runtime representative?

No! (the actual runtime will almost always be faster)

But what can we say about runtime?
QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime ($T(n)$)?
QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime \( T(n) \)?

\[
\begin{align*}
T(0) + T(n - 1) + \Theta(n) & \quad \text{if } X = 1 \\
T(1) + T(n - 2) + \Theta(n) & \quad \text{if } X = 2 \\
T(2) + T(n - 3) + \Theta(n) & \quad \text{if } X = 3 \\
\vdots \\
T(n - 2) + T(1) + \Theta(n) & \quad \text{if } X = n - 1 \\
T(n - 1) + T(0) + \Theta(n) & \quad \text{if } X = n
\end{align*}
\]
How likely are we to pick $X = k$ for any specific $k$?
Probabilities

How likely are we to pick $X = k$ for any specific $k$?

$$P[X = k] = \frac{1}{n}$$
If I roll a d6 (6-sided die) \(k\) times, what is the average roll over all possible outcomes?
If I rolled a d6 1 time... 

<table>
<thead>
<tr>
<th>Roll</th>
<th>Probability</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>1</td>
</tr>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>2</td>
</tr>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>3</td>
</tr>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>4</td>
</tr>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>5</td>
</tr>
<tr>
<td>⚀</td>
<td>1/6</td>
<td>6</td>
</tr>
</tbody>
</table>
Hypothesis: $E[T(n)] \in O(n \log(n))$
Base Case: $E[T(1)] \leq c \log(1)$
Base Case: $E[T(1)] \leq c (1 \log(1))$

$E[T(1)] \leq c (1 \cdot 0)$
Base Case: $E[T(1)] \leq c (1 \log(1))$

$E[T(1)] \leq c (1 \cdot 0)$

$E[T(1)] \leq 0$
Base Case (Take Two): $E[T(2)] \leq c (2 \log(2))$
Base Case (Take Two): $E[T(2)] \leq c \left(2 \log(2)\right)$

$2 \cdot E_{i}[T(i - 1)] + 2c_1 \leq 2c$
Base Case (Take Two): $E[T(2)] \leq c (2 \log(2))$

$2 \cdot E_i[T(i-1)] + 2c_1 \leq 2c$

$2 \cdot (T(0)/2 + T(1)/2) + 2c_1 \leq 2c$
Base Case (Take Two): $E[T(2)] \leq c \cdot (2 \log(2))$

\[
2 \cdot E_i[T(i - 1)] + 2c_1 \leq 2c
\]

\[
2 \cdot (T(0)/2 + T(1)/2) + 2c_1 \leq 2c
\]

\[
T(0) + T(1) + 2c_1 \leq 2c
\]
Base Case (Take Two):

\[ E[T(2)] \leq c \cdot (2 \log(2)) \]

\[ 2 \cdot E_i[T(i - 1)] + 2c_1 \leq 2c \]

\[ 2 \cdot (T(0)/2 + T(1)/2) + 2c_1 \leq 2c \]

\[ T(0) + T(1) + 2c_1 \leq 2c \]

\[ 2c_0 + 2c_1 \leq 2c \]
Base Case (Take Two): $E[T(2)] \leq c \left(2 \log(2)\right)$

$$2 \cdot E_i[T(i - 1)] + 2c_1 \leq 2c$$

$$2 \cdot \left(\frac{T(0)}{2} + \frac{T(1)}{2}\right) + 2c_1 \leq 2c$$

$$T(0) + T(1) + 2c_1 \leq 2c$$

$$2c_0 + 2c_1 \leq 2c$$

True for any $c \geq c_0 + c_1$
Inductive Case

Assume: $E[T(n')] \leq c \ (n' \log(n'))$ for all $n' < n$

Show: $E[T(n)] \leq c \ (n \log(n))$
**Inductive Case**

Assume: \( E[T(n')] \leq c \ (n' \log(n')) \) for all \( n' < n \)

Show: \( E[T(n)] \leq c \ (n \log(n)) \)

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} E[T(i)] \right) + c_1 \leq c n \log(n)
\]
Inductive Case

Assume: $E[T(n')] \leq c \ (n' \log(n'))$ for all $n' < n$

Show: $E[T(n)] \leq c \ (n \log(n))$

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} E[T(i)] \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} ci \log(i) \right) + c_1 \leq cn \log(n)
\]
Inductive Case

Assume: $E[T(n')] \leq c \ (n' \log(n'))$ for all $n' < n$

Show: $E[T(n)] \leq c \ (n \log(n))$

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} E[T(i)] \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} ci \log(i) \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n)
\]
Inductive Case

$$\frac{2}{n} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n)$$
Inductive Case

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2 \log(n)}{n} \left( \sum_{i=0}^{n-1} i \right) + c_1 \leq cn \log(n)
\]
Inductive Case

\[
\frac{2}{n} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2 \log(n)}{n} \left( \sum_{i=0}^{n-1} i \right) + c_1 \leq cn \log(n)
\]

\[
\frac{2 \log(n)}{n} \left( \frac{(n-1)(n-1+1)}{2} \right) + c_1 \leq cn \log(n)
\]
Inductive Case

\[ \frac{2}{n} \sum_{i=0}^{n-1} i \log(n) + c_1 \leq cn \log(n) \]

\[ \frac{2 \log(n)}{n} \sum_{i=0}^{n-1} i + c_1 \leq cn \log(n) \]

\[ \frac{2 \log(n)}{n} \left( \frac{(n-1)(n-1+1)}{2} \right) + c_1 \leq cn \log(n) \]

\[ \frac{\log(n)}{n} (n^2 - n) + c_1 \leq cn \log(n) \]
Inductive Case

\[ \frac{2}{c} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n) \]

\[ c \left( \sum_{i=0}^{n-1} i \right) \leq cn \log(n) \]

\[ c \frac{2 \log(n)}{n} \left( \frac{(n - 1)(n - 1 + 1)}{2} \right) + c_1 \leq cn \log(n) \]

\[ c \frac{\log(n)}{n} (n^2 - n) + c_1 \leq cn \log(n) \]

\[ cn \log(n) - c \log(n) + c_1 \leq cn \log(n) \]
Inductive Case

\[ \frac{2}{n} \left( \sum_{i=0}^{n-1} i \log(n) \right) + c_1 \leq cn \log(n) \]

\[ \frac{2 \log(n)}{n} \left( \sum_{i=0}^{n-1} i \right) + c_1 \leq cn \log(n) \]

\[ \frac{2 \log(n)}{n} \left( \frac{(n-1)(n-1+1)}{2} \right) + c_1 \leq cn \log(n) \]

\[ \frac{\log(n)}{n} (n^2 - n) + c_1 \leq cn \log(n) \]

\[ cn \log(n) - c \log(n) + c_1 \leq cn \log(n) \]

\[ c_1 \leq c \log(n) \]
QuickSort

So...is QuickSort $O(n \log(n))$...?

No!
What guarantees do you get?

If \( f(n) \) is a **Tight Bound**
   The algorithm always runs in \( cf(n) \) steps

If \( f(n) \) is a **Worst-Case Bound**
   The algorithm always runs in at most \( cf(n) \)

If \( f(n) \) is an **Amortized Worst-Case Bound**
   \( n \) invocations of the algorithm **always** run in \( cnf(n) \) steps

If \( f(n) \) is an **Average Bound**
   ...we don't have any guarantees