CSE 250
Data Structures

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Day 12
Inductive Proofs, Divide and Conquer
Textbook Ch. 15
Announcements

- WA1 is Due Wednesday @ 11:59pm
Recap

- **Recursion:** A big problem made up of one or more instances of a smaller problem
  - Factorial: $f(n) = n \times f(n-1)$
  - Fibonacci: $f(n) = f(n-1) + f(n-2)$
  - Towers of Hanoi: $\text{move}(n) = \text{move}(n-1), \text{move}(1), \text{then move}(n-1) \text{ again}$

- **Inductive Proofs:**
  - Come up with a hypothesis
  - Prove it on the base case
  - Assume it works for $n' < n$; Prove for $n$ based on that assumption
Inductive Proof for Towers of Hanoi

• Base case is one ring. I can move one ring.
• Assume I can move n-1 rings; Can I prove that I can move n? Yes
  ○ Move n - 1 (which we can do based on our assumption)
  ○ Move 1 ring
  ○ Move n - 1 (which we can do based on our assumption.
  ○ Therefore, if we can move n - 1, we can move n.
What is the complexity of $\text{fib}(n)$?

def fib(n: Int): Long =
    if(n < 2){ 1 } 
    else { fibb(n-1) + fibb(n-2) }
Fibonacci

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n < 2 \\
T(n - 1) + T(n - 2) + \Theta(1) & \text{otherwise}
\end{cases} \]

Solve for \( T(n) \)...How?
Divide and Conquer

Remember the Towers of Hanoi...
Divide and Conquer

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1. You can move $n$ blocks if you know how to move $n-1$ blocks
Divide and Conquer

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1. You can move \( n \) blocks if you know how to move \( n-1 \) blocks
2. You can move \( n-1 \) blocks if you know how to move \( n-2 \) blocks
3. You can move \( n-2 \) blocks if you know how to move \( n-3 \) blocks
Divide and Conquer

Remember the Towers of Hanoi...

1. You can move $n$ blocks if you know how to move $n-1$ blocks
2. You can move $n-1$ blocks if you know how to move $n-2$ blocks
3. You can move $n-2$ blocks if you know how to move $n-3$ blocks
4. You can move $n-3$ blocks if you know how to move $n-4$ blocks
Divide and Conquer

Remember the Towers of Hanoi...

1. You can move \( n \) blocks if you know how to move \( n-1 \) blocks
2. You can move \( n-1 \) blocks if you know how to move \( n-2 \) blocks
3. You can move \( n-2 \) blocks if you know how to move \( n-3 \) blocks
4. You can move \( n-3 \) blocks if you know how to move \( n-4 \) blocks

... 

You can always move 1 block
Divide and Conquer

To solve the problem at $n$: 
Divide and Conquer

To solve the problem at $n$:

**Divide** the problem into smaller problems (size $n-1$ and 1 in this case)
Divide and Conquer

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**Conquer** the smaller problems
Divide and Conquer

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**Divide** the problem into smaller problems (size $n-1$ and 1 in this case)

**Conquer** the smaller problems

**Combine** the smaller solutions to get the bigger solution
Merge Sort

**Input:** An array with elements in an unknown order.

**Output:** An array with elements in sorted order.
Merge Sort - Questions

**Divide** (break the array into smaller arrays)
What's the smallest list I could try to sort?
**Divide** (break the array into smaller arrays)

What's the smallest list I could try to sort? size $n = 1$
Divide (break the array into smaller arrays)
What's the smallest list I could try to sort? size n = 1

Conquer (sort the smaller arrays)
How do I sort it?
Merge Sort - Questions

**Divide** (break the array into smaller arrays)
What's the smallest list I could try to sort? size $n = 1$

**Conquer** (sort the smaller arrays)
How do I sort it? It's already sorted!!!
Merge Sort - Questions

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How do I sort it? It's already sorted!!!

**Combine** (combine the sorted arrays into a bigger sorted array)
How can I do this, and how long does it take?
Merge Sort - Questions

**Divide** (break the array into smaller arrays)
What's the smallest list I could try to sort? size n = 1

**Conquer** (sort the smaller arrays)
How do I sort it? It's already sorted!!!

**Combine** (combine the sorted arrays into a bigger sorted array)
How can I do this, and how long does it take? Merge...
How do we Merge Two Sorted Arrays?

24  37  62  73  95
15  31  55  61  88
How do we Merge Two Sorted Arrays?

24 37 62 73 95

31 55 61 88

15
How do we Merge Two Sorted Arrays?
How do we Merge Two Sorted Arrays?
How do we Merge Two Sorted Arrays?
How do we Merge Two Sorted Arrays?

15 24 31 37 55

62 73 95

61 88
How do we Merge Two Sorted Arrays?
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15 24 31 37 55 61 62

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What was the complexity?

15  24  31  37  55  61  62  73  88  95
How do we Merge Two Sorted Arrays?

What was the complexity?

Each comparison was $\Theta(1)$...
How do we Merge Two Sorted Arrays?

What was the complexity?

Each comparison was $\Theta(1)$...

How many comparisons? $\Theta(|\text{red}| + |\text{blue}|)$
def merge[A: Ordering](left: Seq[A], right: Seq[A]): Seq[A] = {
  val output = ArrayBuffer[A]()

  val leftItems = left.iterator.buffered
  val rightItems = right.iterator.buffered

  while(leftItems.hasNext || rightItems.hasNext) {
    if(!left.hasNext)          { output.append(right.next) }
    else if(!right.hasNext)    { output.append(left.next) }
    else if(Ordering[A].lt( left.head, right.head ))
      { output.append(left.next) }
    else
      { output.append(right.next) }
  }

  output.toSeq
}
Divide

- We know how to combine sorted arrays
- We know that in a based case of $n = 1$ how to sort
- How do we divide our problem to get there?
Divide

- We know how to combine sorted arrays
- We know that in a base case of $n = 1$ how to sort
- How do we divide our problem to get there?

Let's divide our array in half (recursively)!
Visualization
Visualization

Divide the array in half
Visualization

Divide the array in half (recursively)
Visualization

Divide the array in half (recursively)

Conquer (sort) each half

Combine (merge) each solution
Visualization

**Divide** the array in half (recursively)

**Conquer** (sort) each half

**Combine** (merge) each solution
Visualization

**Divide** the array in half (recursively)
**Conquer** (sort) each half
**Combine** (merge) each solution
Visualization

Divide the array in half
Conquer (sort) each half
Combine (merge) each half
def sort[A: Ordering](data: Seq[A]): Seq[A] = 
{
    if(data.length <= 1) { return data }
    else {
        val (left, right) = data.splitAt(data.length / 2)
        return merge(
            sort(left),
            sort(right)
        )
    }
}
Complexity

If we solve a problem of size $n$ by:

- Dividing it into $a$ sub-problems
  - Where each problem is of size $n/b$ (usually $b = a$)
  - ...and stop recurring at $n \leq c$
  - ...and the cost of dividing is $D(n)$
  - ...and the cost of combining is $C(n)$

Then our total cost will be...
Complexity

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq c \\
 a \cdot T\left(\frac{n}{b}\right) + D(n) + C(n) & \text{otherwise} 
\end{cases} \]

\begin{align*}
\text{a subproblems of size } n/b, \text{ base case of } n \leq c \\
\text{divide cost of } D(n) \\
\text{and combine cost of } C(n)
\end{align*}
For Merge Sort

**Divide:** Split the sequence in half

\[ D(n) = \Theta(n) \text{ (can we do it faster?)} \]

**Conquer:** Sort left and right halves

\[ a = 2, \quad b = 2, \quad c = 1 \]

**Combine:** Merge halves together

\[ C(n) = \Theta(n) \]
For Merge Sort

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases} \]
For Merge Sort

For $n \leq 1$:

$$T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2 \cdot T(n/2) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases}$$

How do we find a closed-form hypothesis?
For Merge Sort: Recursion Trees

\[
T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases}
\]
For Merge Sort: Recursion Trees

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases} \]

Each node shows \( D(n) + C(n) \)
For Merge Sort: Recursion Trees

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) + \Theta(n) & \text{otherwise}
\end{cases} \]

Each node shows \( D(n) + C(n) \)

log(n) height
For Merge Sort: Recursion Trees

At level $i$ there are $2^i$ tasks, each with runtime $\Theta(n/2^i)$, and there are $\log(n)$ levels.
For Merge Sort: Recursion Trees

At level $i$ there are $2^i$ tasks, each with runtime $\Theta(n/2^i)$, and there are $\log(n)$ levels.

$$
\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
$$
For Merge Sort: Recursion Trees

At level $i$ there are $2^i$ tasks, each with runtime $\Theta(n/2^i)$, and there are $\log(n)$ levels.

$$\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)$$
At level $i$ there are $2^i$ tasks, each with runtime $\Theta(n/2^i)$, and there are $\log(n)$ levels.
For Merge Sort: Recursion Trees

At level $i$ there are $2^i$ tasks, each with runtime $\Theta(n/2^i)$, and there are $\log(n)$ levels.
Merge Sort Runtime

\[
\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
\]
Merge Sort Runtime

\[
\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
\]

\[
\sum_{i=0}^{\log(n)} (2^i + 1 - 1)\Theta\left(\frac{n}{2^i}\right)
\]
Merge Sort Runtime

\[
\log(n) \sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
\]

\[
\log(n) \sum_{i=0}^{\log(n)} (2^i + 1 - 1) \Theta\left(\frac{n}{2^i}\right)
\]

\[
\log(n) \sum_{i=0}^{\log(n)} 2^i \Theta\left(\frac{n}{2^i}\right)
\]
Merge Sort Runtime

\[ \sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta \left( \frac{n}{2^i} \right) \]
Merge Sort Runtime

\[ \log(n) \sum_{i=0}^{\log(n)} 2^i \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right) \]

\[ \log(n) \sum_{i=0}^{\log(n)} \Theta(n) \]
Merge Sort Runtime

\[
\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
\]

\[
\sum_{i=0}^{\log(n)} \Theta(n)
\]

\[
(\log(n) - 0 + 1)\Theta(n)
\]
Merge Sort Runtime

\[
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\[
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\]

\[
(log(n) - 0 + 1)\Theta(n)
\]

\[
\Theta(n \log(n)) + \Theta(n)
\]
Merge Sort Runtime

\[
\log(n) \sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta\left(\frac{n}{2^i}\right)
\]

\[
\sum_{i=0}^{\log(n)} \Theta(n)
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(\log(n) - 0 + 1)\Theta(n)
\]

\[
\Theta(n \log(n)) + \Theta(n)
\]

\[
\Theta(n \log(n))
\]
Now we can use induction to prove that there is a $c, n_0$ s.t. $T(n) \leq c \ n \log(n)$ for any $n > n_0$.
Base Case: $T(1) \leq c$

$c_0 \leq c$

True for any $c > c_0$
Assume: $T(n/2) \leq c (n/2) \log(n/2)$

Show: $T(n) \leq cn \log(n)$
Merge Sort Runtime: Inductive Proof

Assume: \( T(n/2) \leq c \frac{n}{2} \log(n/2) \)

Show: \( T(n) \leq cn \log(n) \)

\[
2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)
\]
Merge Sort Runtime: Inductive Proof

**Assume:** $T(n/2) \leq c (n/2) \log(n/2)$

**Show:** $T(n) \leq cn \log(n)$

$$2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)$$

By the assumption, and transitivity, we just need to show:

$$2c \frac{n}{2} \log\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)$$
Assume: $T(n/2) \leq c (n/2) \log(n/2)$

Show: $T(n) \leq cn \log(n)$

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By the assumption, and transitivity, we just need to show:

$$2c\frac{n}{2} \log\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)$$

$$cn \log(n) - cn \log(2) + c_1 + c_2n \leq cn \log(n)$$
Assume: $T(n/2) \leq c \left(\frac{n}{2}\right) \log\left(\frac{n}{2}\right)$

Show: $T(n) \leq cn \log(n)$

$$2 \cdot T\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)$$

By the assumption, and transitivity, we just need to show:

$$2c \frac{n}{2} \log\left(\frac{n}{2}\right) + c_1 + c_2n \leq cn \log(n)$$

$$cn \log(n) - cn \log(2) + c_1 + c_2n \leq cn \log(n)$$

$$c_1 + c_2n \leq cn \log(2)$$
Merge Sort Runtime: Inductive Proof

\[ c_1 + c_2 n \leq cn \log(2) \]
Merge Sort Runtime: Inductive Proof

\[ c_1 + c_2 n \leq c n \log(2) \]

\[ \frac{c_1}{n \log(2)} + \frac{c_2}{\log(2)} \leq c \]
**Merge Sort Runtime: Inductive Proof**

\[
c_1 + c_2 n \leq c n \log(2)
\]

\[
\frac{c_1}{n \log(2)} + \frac{c_2}{\log(2)} \leq c
\]

Which is true for any

\[
n_0 \geq \frac{c_1}{\log(2)} \quad \text{and} \quad c > \frac{c_2}{\log(2)} + 1
\]
Next Time...

Quick Sort
Average Runtime