CSE 250
Lecture 38

Final Review

Day 2

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Edge List Summary

- addEdge, addVertex: **O(1)**
- removeEdge: **O(1)**
- removeVertex: **O(1) + O(vertex.incidentEdges)**
- vertex.outEdges, vertex.inEdges, vertex.incidentEdges: **O(m)**
  - (total cost to visit all out/in/incident edges)
- vertex.edgeTo: **O(m)**
- **Space Used**: **O(n+m)**

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Add an Adjacency List

```
class DirectedGraphV3[LV, LE]
{
  def addEdge(orig: Vertex, dest: Vertex, label: LE): Edge =
  {
    val edge = new Edge(label)
    edge._listNode = edges.append(edge)
    orig._outEdges.append(edge)
    dest._inEdges.append(edge)
    return edge
  }
  class Vertex(_label: LV){
    val _outEdges: LinkedList[Edge]
    val _inEdges: LinkedList[Edge]
    // …
  }
}
```


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Adjacency List Summary

- addEdge, addVertex: **O(1)**
- removeEdge: **O(1)**
- removeVertex: **O(deg(vertex))**
- vertex.outEdges: **O(|outEdges|)** to visit all outEdges
  - Same for vertex.inEdges, vertex.incidentEdges
- vertex.edgeTo: **O(|outEdges|)**
- **Space Used**: **O(n+m)**

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Binary Search Trees

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Tree Terminology

- Rooted directed tree
  - **root** is the topmost vertex
  - EmptyTree contains 0 vertices, null for mutable tree.
- **Parent** references one or more **child**ren
  - **leaf** vertex: Vertex with zero children
- **Depth** of a vertex
  - Number of edges in the path from the root to the vertex
- **Level** of a vertex
  - Depth + 1

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Tree Terminology

- The **size** of a tree
  - the number of vertices
  - Typically represented as **n**
- The **depth** of a tree - the maximum depth of any node
  - Typically represented as **d**
- The **height** of a vertex
  - The maximum number of edges from vertex to any leaf

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Tree Terminology

- A **binary tree** is a tree where
  - every vertex has ≤2 children
- A **full binary tree** is a tree where
  - all leaf vertices are at the lowest depth of the tree
  - Every vertex has either 0 or 2 children
- Depth of a full tree: d
- Size of a full tree: $n = \sum_{i=0}^{d} 2^i = 2^{d+1} - 1 = O(2^d)$

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Tree Traversals

- Pre-order (top-down)
  - visit **root**, visit **left** subtree, visit **right** subtree
- In-order
  - visit **left** subtree, visit **root**, visit **right** subtree
- Post-order (bottom-up)
  - visit **left** subtree, visit **right** subtree, visit **root**

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Computing the height of a tree

- Height (depth) of a tree = height of the root

$$d(\texttt{root}) = \begin{cases} -1 & \textbf{if the tree is empty} \\ 1 + max(d(\texttt{root.left}), d(\texttt{root.right})) & \textbf{otherwise} \end{cases}$$

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Priority Queues / Heaps

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Priority Queue

- PriorityQueue[A: Ordering]
  - enqueue(v: A): Unit
    - Insert value v into the priority queue
  - head: A
    - Retrieve the highest-priority value in the priority queue
  - dequeue: A
    - Remove the highest-priority value from the priority queue

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(Binary) Heap

- **Idea**: Keep the priority queue “kinda” sorted
  - Keep larger items closer to the front of the list
  - Trade off between...
    - Moving larger elements forward
    - Leaving some elements out-of-order
- **Challenge**: How track which elements are already sorted?
- **Inspiration**: Trees

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(Binary) Heaps

- A (binary) heap is a tree-like structure with the properties:
  - A complete (binary) tree
  - Each vertex is “non-increasing” relative to its children
    - Strictly decreasing if no duplicates present
- A complete (binary) tree is a tree where
  - Each node has at most 2 children
  - Every level except for the last is full
    - Nodes in the last level are as far left as possible

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Heaps

- What is the max depth of a binary heap?
  - Level 1: 1 value
  - Level 2: up to 2 values
  - Level 3: up to 4 values
  - Level 4: up to 8 values
  - Level i: up to 2^i values

$$n = \sum_{i=1}^{\ell_{max}} 2^i = 2^{\ell_{max}+1} - 1 \qquad \ell_{max} = \log(n+1) - 1$$

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Heap Methods

- isEmpty: Boolean
- length: Int
- head: A
- pushHeap(elem: A)
- popHeap: A

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Heap Methods: pushHeap

- **Idea**: Insert into the next available location and then fix up
  - Insert at next available location (call it **`current`**)
  - While **`current`** isn’t **`root`** and **`parent < current`**
    - Swap **`current`** and **`parent`**
    - Repeat with **`current = parent`**

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Heap Methods: popHeap

- **Idea**: Fill root with value in last filled location and then fix down
  - Start with the root (call it **`current`**)
  - While **`current`** isn't a leaf and there's a **`child < current`**
    - Swap **`current`** and the larger **`child`**
    - Repeat with **`current = child`**

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Storing Heaps in Memory

- **Observations**:
  - Each layer has a maximum size
  - Each layer grows left-to-right
  - Only the last layer grows
- **Idea**: Use an array to store the heap

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Analysis

- pushHeap
  - Append to end of ArrayBuffer
    - Amortized O(1)
  - fixUp
    - log(n) steps, each O(1) = O(log(n))

O(log(n)) amortized
O(n) worst-case

- popHeap
  - Remove end of ArrayBuffer
    - O(1)
  - fixDown
    - log(n) steps, each O(1) = O(log(n))

O(log(n))

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Binary Search Trees

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Binary Search Tree

- Store key/value pairs (T = (K, V))
  - Require an Ordering[K]
- Enforce constraints:
  - No duplicate keys
  - For every vertex v_L in the left subtree of v_1,
    - v_L.key < v_1.key
  - For every vertex v_R in the right subtree of v_1,
    - v_R.key > v_1.key

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BST Mutations

Operation	Runtime
find	O(d)
insert	O(d)
remove	O(d)

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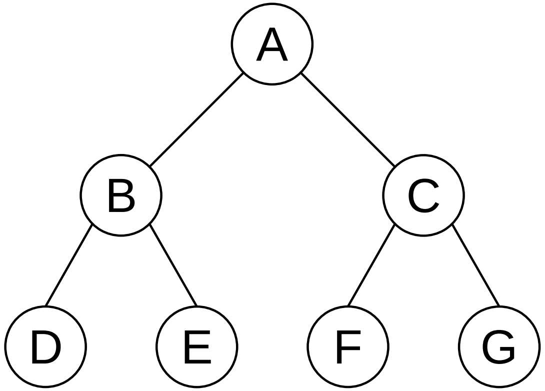
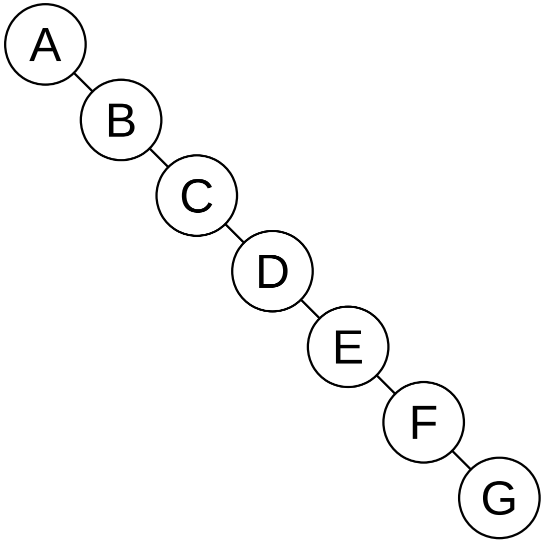
Tree Depth vs Size

height(left) ≈ height(right)



d = O(log(n))

height(left) ⋘ height(right)



d = O(n)

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“Balanced” Trees

- Faster search: Want height(left) ≈ height(right)
  - Make it more precise: |height(left) - height(right)| ≤ 1
  - (left, right height differ by at most 1)
- **Question**: How do we keep the tree balanced?
  - Option 1: Keep left/right subtrees within **+/- 1** of each other
    - Add a field to track the “imbalance factor”
  - Option 2: Ensure leaves are at a minimum depth of **d / 2**
    - Add a designation marking each node as red or black

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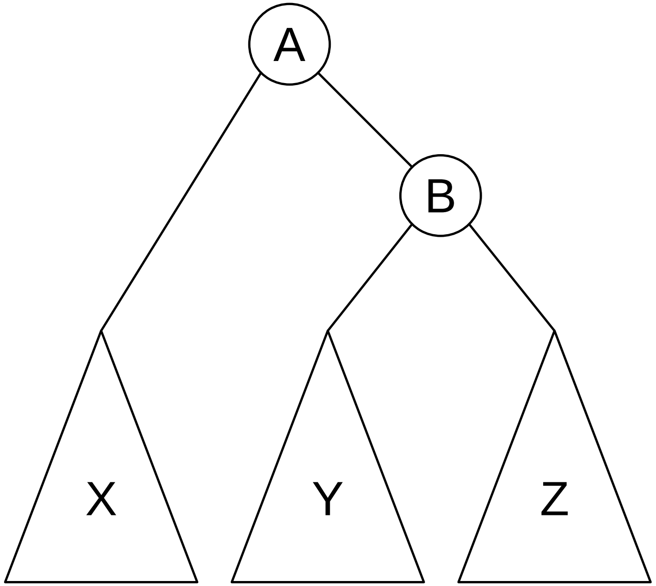
Rebalancing Trees



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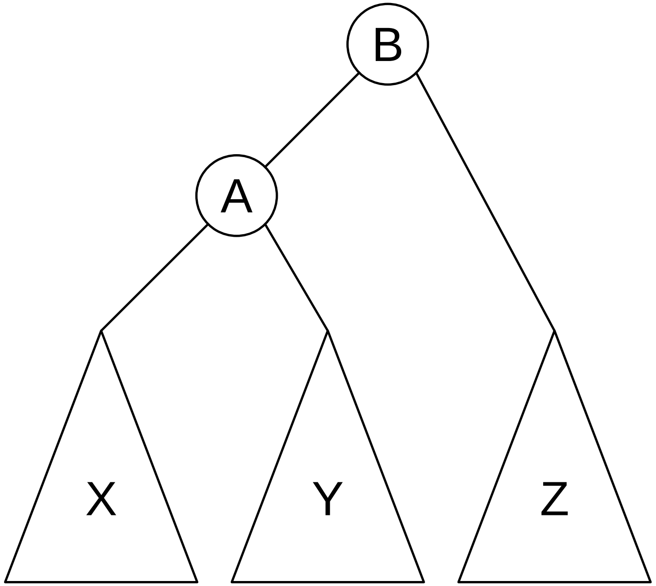
Rebalancing Trees



Rotate(A, B)

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AVL Trees

- An AVL tree (Adelson-Velsky and Landis) is a BST where every node is "depth-balanced"
  - |depth(left subtree) - depth(right subtree)| < 1
- define **`balance(v) = height(v.right) - height(v.left)`**
  - Maintain balance(v) ∈ { -1, 0, 1 }
    - balance(v) = 0 → "v is balanced"
    - balance(v) = -1 → "v is left-heavy"
    - balance(v) = 1 → "v is right-heavy"

If the balance constraint is obeyed, the tree must have $\Omega(2^d)$ nodes (d = log(n))

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Maintaining Balance

- Enforcing height-balance is too strict
  - May require “unnecessary” rotations
- Weaker restriction:
  - Balance the depth of EmptyTree nodes
  - If a, b are EmptyTree nodes:
    - depth(a) $\geq$ (depth(b) $\div$ 2)

      or
    - depth(b) $\geq$ (depth(a) $\div$ 2)

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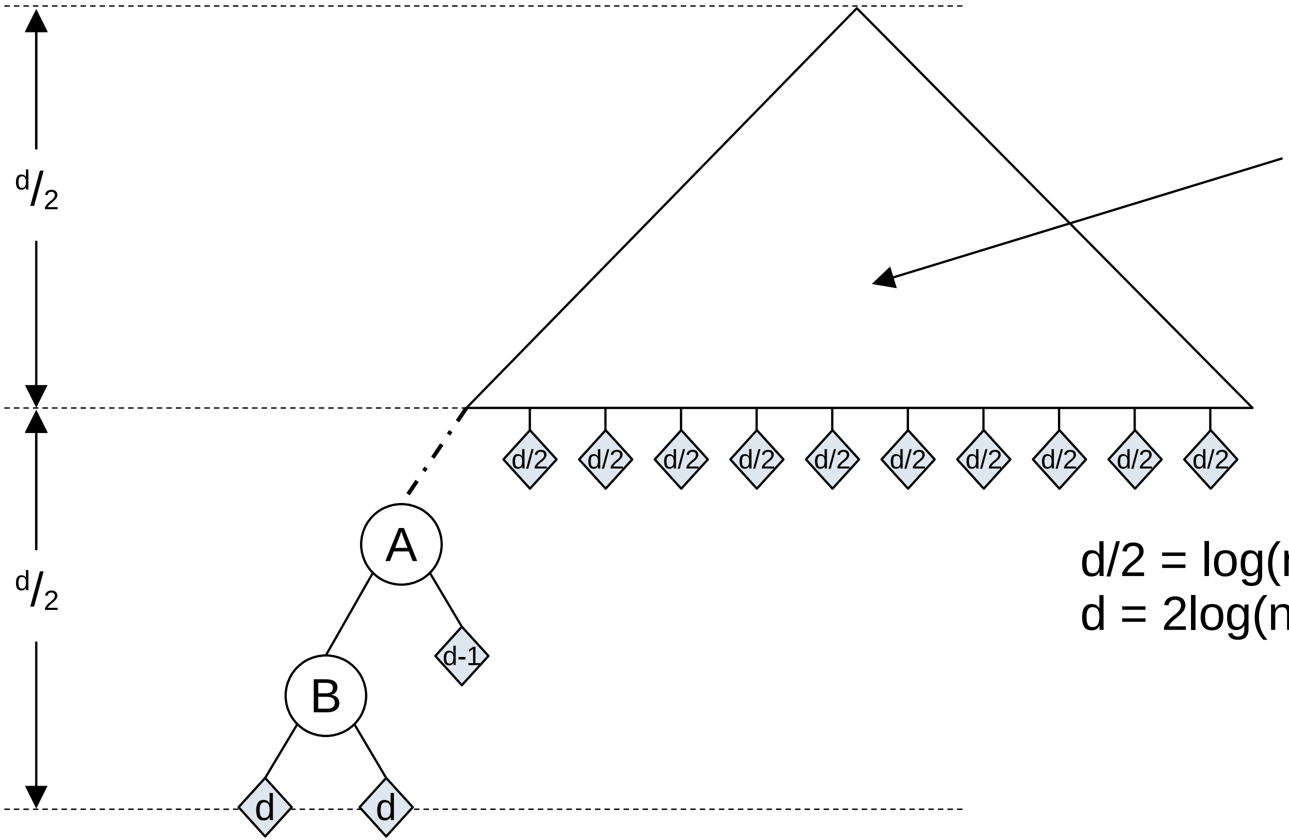
Balancing Empty Node Depth



Must be full
($2^{\lceil d/2 \rceil}$ nodes)

$d/2 = \log(n)$
$d = 2\log(n) = O(\log(n))$

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Red-Black Trees

- Color each node red or black
  1) # of black nodes from each empty to root must be identical
  2) Parent of a red node must be black

- On Insertion (or deletion)
  - Inserted node is red (won’t change # of black nodes)
  - “Repair” violations of rule 2 by rotating or recoloring
    - Each repair guarantees rule 1 is preserved
    - Each repair creates at most 1 new violation of rule 2 at the parent.

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TreeSet[A: Ordering]

- **add(a: A): Unit**
  - O(log(n)) – Insert **a** into the balanced binary search tree
- **apply(a: A): Boolean**
  - O(log(n)) – Find **a** in the binary search tree, return true if found
- **remove(a: A): Unit**
  - O(log(n)) – Remove **a** from the binary search tree

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TreeMap[K: Ordering, V]

- **put(k: K, v: V): Unit**

O(log(n)) – Insert the pair (**k**,**v**) into the balanced binary search tree according to the ordering on **k**.

- **apply(k: K): V**

O(log(n)) – Find **k** in the binary search tree, return the matching **v**.

- **remove(k: K): Unit**

O(log(n)) – Remove **k** from the binary search tree.

- **range(from: K, until: K): TreeMap[K, V]**
  - Return a sub-map containing only keys in the range [from,until)

O(log(n)+|range|)

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Hash Tables

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Hash Table with Chaining

- Create an array of size N
- Pick an O(1) function h(k) to assign each record to [0,N)
  - A record with key k can only be stored in bucket h(k)
  - Use linked lists if the bin is occupied

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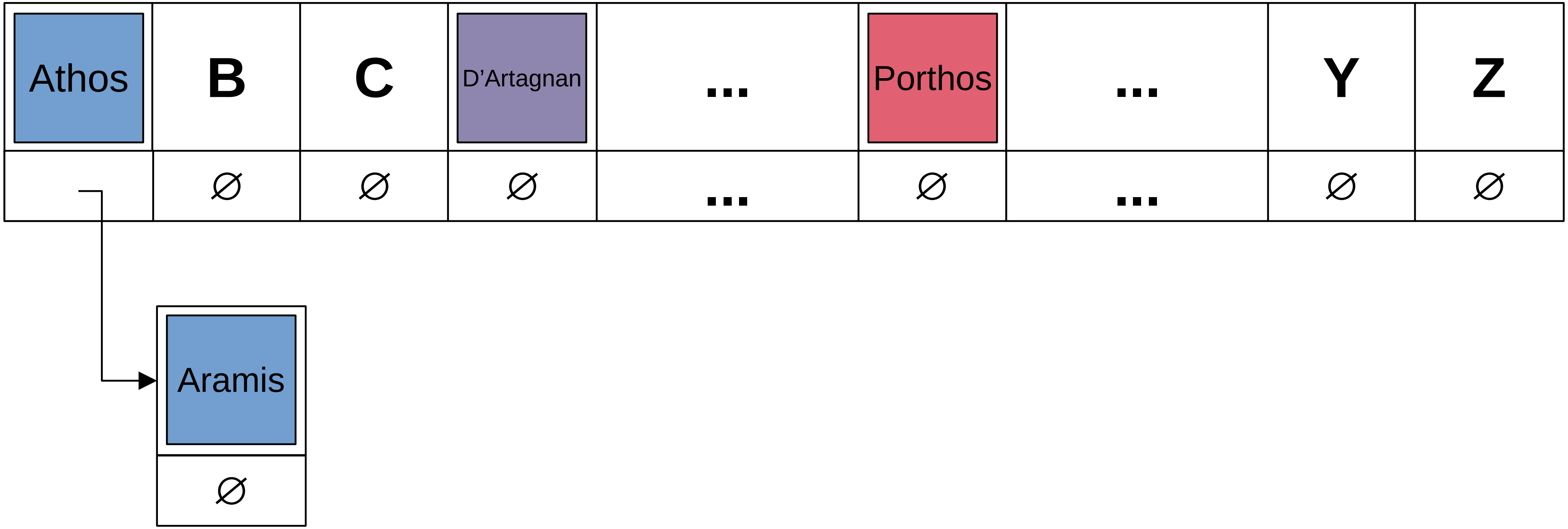
Hash Table with Chaining



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Picking a Lookup Function

- Desirable Features for h(x)
  - Fast
    - needs to be O(1)
  - “Unique”
    - As few duplicate bins as possible

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Hash Functions

- Examples
  - SHA256 ← used by GIT
  - MD5, BCRYPT ← used by unix login, apt
  - MurmurHash3 ← used by Scala
- hash(x) is pseudorandom
  1) hash(x) ~ uniform random value in [0, INT_MAX)
  2) hash(x) always returns the same value
  3) hash(x) uncorrelated with hash(y) for $x \neq y$

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Lookup Table

- We want fewer than INT_MAX buckets
- Store a record with key k in bucket h(k) % N

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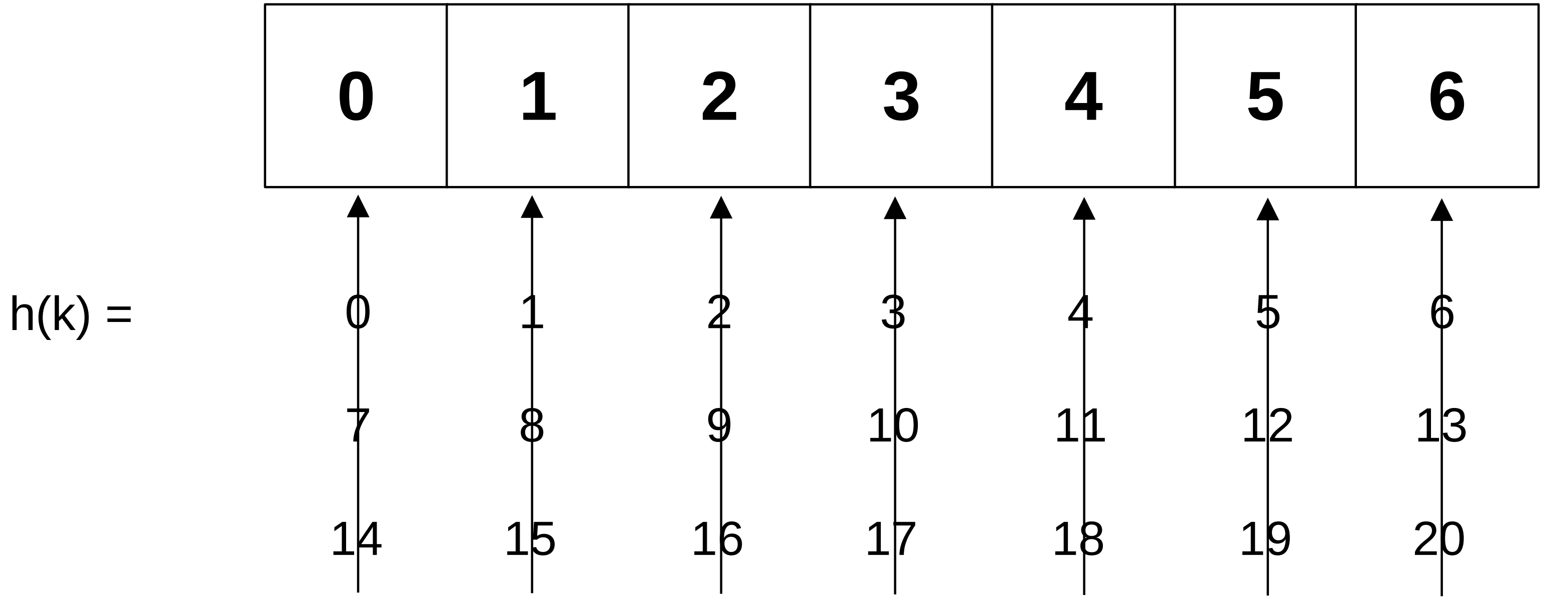
Modulus



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Iterating over a hash table

- Runtime
  - Visit every hash bucket
    - $O(N)$
  - Visit every element in every bucket
    - $O(n)$

    $= O(N + n)$

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Hash Functions + Buckets

Everything is: $O\left(\frac{n}{N}\right)$ Let's call $\alpha = \frac{n}{N}$ the load factor.

Idea: Make α a constant

Fix an α_{max} and start requiring that $\alpha \leq \alpha_{max}$

What happens when the user inserts $n = N \times \alpha_{max} + 1$ records ?

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Rehashing

- Resize the array from N_{old} to N_{new}.
  - Element x moves from hash(x) % N_{old} to hash(x) % N_{new}
- Runtime?
  - Allocate new array: **O(1)**
  - Visit every hash bucket: **O(N_{old})**
  - Hash and copy each element to the new array: **O(n)**
  - Free the old array: **O(1)**
  - $O(1) + O(N_{old}) + O(n) + O(1) = O(N_{old}+n)$

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Rehashing

- Whenever $\alpha > \alpha_{max}$, rehash to double size
  - Contrast with ArrayBuffer
- Starting with <u>N</u> buckets, after <u>n</u> insertions..
  - Rehash at $n_1 = \alpha_{max} \times N$: From N to 2N Buckets
  - Rehash at $n_2 = \alpha_{max} \times 2N$: From 2N to 4N Buckets
  - Rehash at $n_3 = \alpha_{max} \times 4N$: From 4N to 8N Buckets
  - ...
  - Rehash at $n_j = \alpha_{max} \times 2^j N$: From $2^{j-1}N$ to $2^j N$ Buckets

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Number of Rehashes

With n insertions...

$$n = 2^j \alpha_{max}$$

$$2^j = \frac{n}{\alpha_{max}}$$

$$j = \log\left(\frac{n}{\alpha_{max}}\right)$$

$$j = \log(n) - \log(\alpha_{max})$$

$$j = O(\log(n))$$

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Total Work

Rehashes required: $O(log(n))$

The i-th rehashing: $O(2^i N)$

Total work after n insertions...

$$\sum_{i=0}^{O(\log(n))} O(2^i N) = O\left(\sum_{i=0}^{O(\log(n))} 2^i + \sum_{i=0}^{O(\log(n))} N\right)$$

$$= O\left(2^{O(\log(n)+1)} - 1 + O(\log(n)N)\right)$$

$$= O\left(n + N\log(n)\right)$$

Work per insertion:
(ammortized cost)

$$O\left(\frac{n + N\log(n)}{n}\right) = O\left(\frac{n}{n} + \frac{N\log(n)}{n}\right) = O(1)$$

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HashSet[A]

- **add(a: A): Unit**

  expected: O(1)
  worst-case: O(N)

  - Compare all elements in bucket **h(a) % N** to **a**. If a match is not present, insert **a** at the head.

- **apply(a: A): Boolean**

  expected: O(1)
  worst-case: O(N)

  - Compare all elements in bucket **h(a) % N** to **a**. If a match is found, return true.

- **remove(a: A): Unit**

  expected: O(1)
  worst-case: O(N)

  - Compare all elements in bucket **h(a) % N** to **a**. If a match is found, remove the matched element.

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HashMap[K, V]

- **put(k: K, v: V): Unit**
  expected: O(1)
  worst-case: O(N)
  - Compare the key of all elements in bucket **h(k) % N** to **k**. If a match is present, remove it. Insert (**k**, **v**) at the head
- **apply(k: K): V**
  expected: O(1)
  worst-case: O(N)
  - Compare the key of all elements in bucket **h(k) % N** to **k**. If a match is found, return the corresponding value.
- **remove(a: A): Unit**
  expected: O(1)
  worst-case: O(N)
  - Compare the key of all elements in bucket **h(k) % N** to **k**. If a match is found, remove the matching element.
- **NO range operation**

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Variations

- **Hash Table with Chaining**
  - … but re-use empty hash buckets instead of chaining
    - **Hash Table with Open Addressing**
    - **Cuckoo Hashing** (Double Hashing)
  - … but avoid bursty rehashing costs
    - **Dynamic Hashing**
  - … but avoid O(N) iteration cost
    - **Linked Hash Table**

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Open Addressing

- insert(X)
  - While bucket hash(X)+i %N is occupied, i = i + 1
  - Insert at bucket hash(X)+i %N
- apply(X)
  - While bucket hash(X)+i %N is occupied
    - If the element at bucket hash(X)+i %N is X, return it
    - Otherwise i = i + 1
  - Element not found

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Open Addressing

- **Linear Probing**: Offset to hash(X)+ci for some constant c
- **Quadratic Probing**: Offset to hash(X)+ci^2 for some constant c
- Follow Probing Strategy to find the next bucket

- Runtime Costs
  - Chaining: Dominated by following chain
  - Open Addressing: Dominated by probing
- With a low enough α_{max}, operations still O(1)

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Cuckoo Hashing

- Use two hash functions: $hash_1$, $hash_2$
  - Each record is stored at one of the two
- insert(x)
  - If both buckets are available: pick at random
  - If one bucket is available: insert record there
  - If neither bucket is available, pick one at random
    - “Displace” the record there, move it to the other bucket
    - Repeat displacement until an empty bucket is found

apply(x) and remove(x) is guaranteed O(1)

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Linked Hash Table

- Iteration over Hash Table is O(N + n)
  - Can be much slower than O(n)
- **Idea**: Connect entries together in a Doubly Linked List

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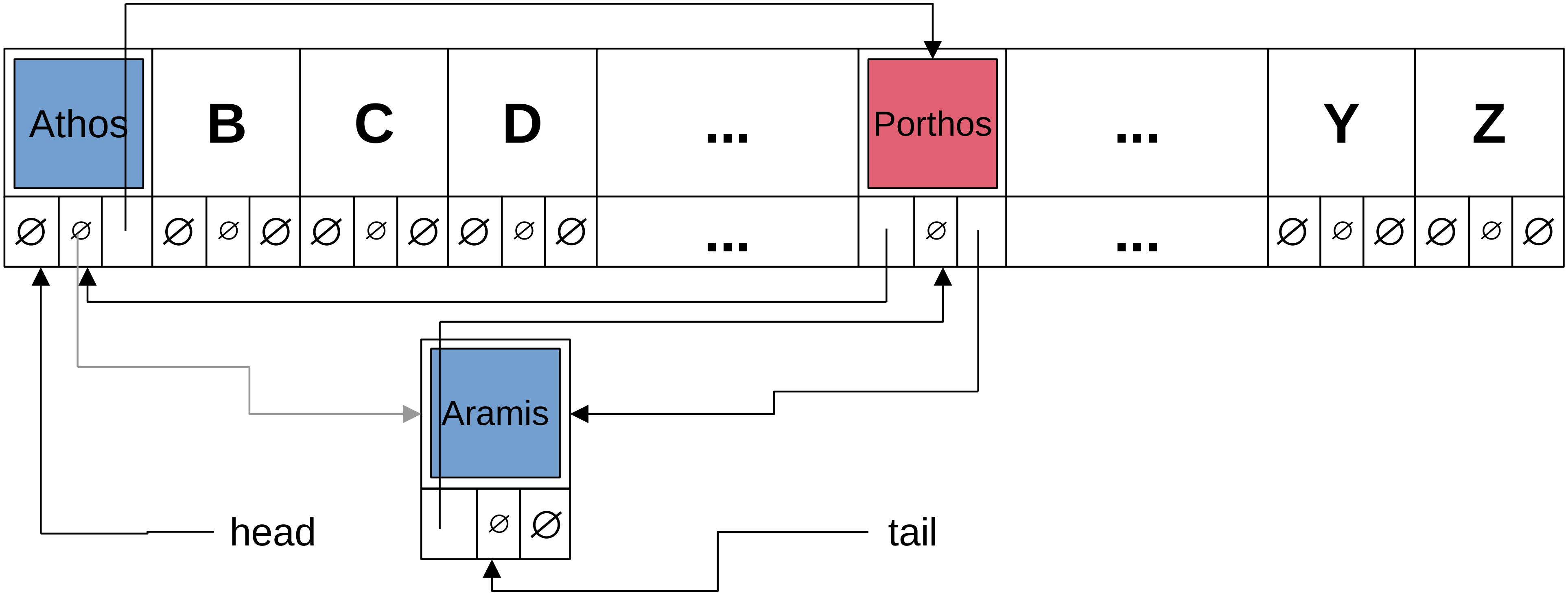
Linked Hash Table



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Linked Hash Table

- O(n) Iteration
- apply(x)
  - O(1) increase in cost
- insert(x)
  - O(1) increase in cost
- remove(x)
  - O(1) increase in cost

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Lossy Sets / Bloom Filters

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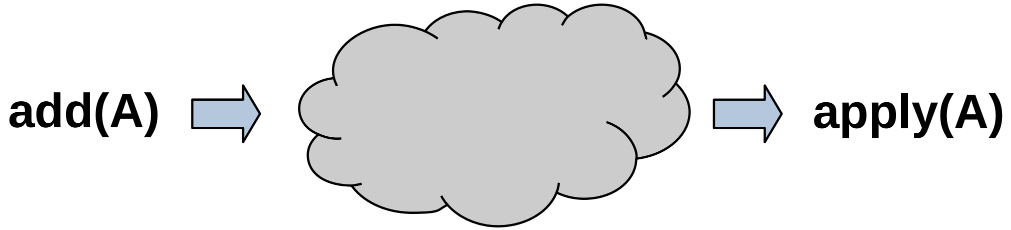
“Lossy Sets”

- Set[A]
  - **add(a: A)**: Insert **a** into the set
  - **apply(a: A)**: Return true if **a** is in the set



- What if we didn’t need apply to be perfect?

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Lossy Sets

- LossySet[A]
  - **add(a: A)**: Insert **a** into the set.
  - **apply(a: A)**:
    - If **a** is in the set, always return true
    - If **a** is not in the set, usually return false
      - Is allowed to return true, even if **a** is not in the set

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Bloom Filters

```scala
class BloomFilter[A](_size: Int, _k: Int) extends LossySet[A]
{
  val bits = new Array[Boolean](_size)

  def add(a: A): Unit = {
    for(i <- 0 until _k) { bits( ithHash(a, i) % _size ) = true }
  }

  def apply(a: A): Boolean = {
    for(i <- 0 until _k) {
      if( !bits( ithHash(a, i) % _size ) { return false; }
    }
    return true
  }
}
```


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Bloom Filter Parameters

- _size
  - Intuitively: More space, fewer collisions
- _k
  - Intuitively: more hash functions means…
    - …more chances for one of **b**’s bits to be unset.
    - …more bits set = higher chance of collisions.

To preserve a constant false-positive rate:
Grow _size as O(n)
Value of _k is fixed for a given size.

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Bloom Filters: Analysis

- N/n = 5 → ~10% collision chance
- N/n = 10 → ~1% collision chance

- 10 bits vs
  - 32 bits for one Int (3 to 1 savings)
  - 64 bits for a Double/Long (6 to 1 savings)
  - ~8000 bits for a full record (800 to 1 savings)

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